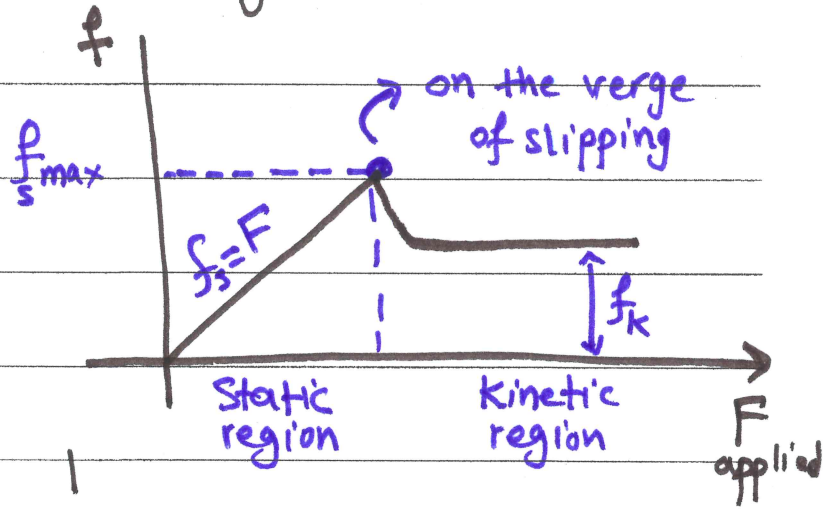
[I] Friction force

■ The basic characteristics of the force of friction are purely empirical, i.e., are based on experimental observations. Here are the findings we would obtain:

(i) $\vec{F}$ ← [m]

If you apply an external force $\vec{F}$ to m and it remains stationary, then according to Newton's 2nd law, the force that counteracts $\vec{F}$ and keeps m at rest acts toward the right and is called static friction force $\vec{f}_s$. The direction of $\vec{f}_s$ is always opposite to that of $\vec{F}$ relative to the surface of contact between the object m and its supporting surface.



∴ $\vec{F}$ ← [m] ∴ $\vec{f}_s$

(ii) If $\vec{F}$ is increased, $\vec{f}_s$ also increases. When m is on the verge of slipping, f_s has its maximum value f_s^{max} as shown above.

In other words, if m is at rest, it takes a certain threshold amount of external force $\vec{F}$ to set it in motion. $\Rightarrow F_{thre.} = f_{s\,max}$.

(iii) When F exceeds $f_{s\,max}$, m accelerates to the left. The friction force in motion is called kinetic friction force $\vec{f}_k$. The direction of $\vec{f}_k$ is always opposite to the direction of motion of m relative to the surface it moves on: $\vec{F} \leftarrow [m] \rightarrow \vec{f}_k$

(iv) Friction force depends on the roughness of the contact surface between m and the supporting surface. Thus both f_s and f_k are proportional to the magnitude of the normal force exerted by the rough contact surface. However, f_k is less than $f_{s\,max}$. For the magnitude of f_s, we can write

$f_s \leq \mu_s n$, $f_{s\,max} = \mu_s n$ ——— (1)

μ_s is the coefficient of static friction. Likewise,

$f_k = \mu_k n$ ——— (2), where μ_k is the coefficient of kinetic friction.

$\Rightarrow f_{s\,max} > f_k \Rightarrow \mu_s > \mu_k$ ——— (3)

Some typical coefficients of static and kinetic friction are shown in Table 4.2 in your text. The case where $\mu = 0$ corresponds to a frictionless approximation (idealized smooth surface).

(v) The net force on m is $[F - f_k]$ in the x-direction, produces an acceleration to the left. If $F = f_k$, $a = 0$ and m can move to the left with constant speed. If $\vec{F}$ is removed from the moving object m, the $\vec{f}_k$ acting to the right provides an acceleration and eventually brings m to rest, according to Newton's 2nd law.

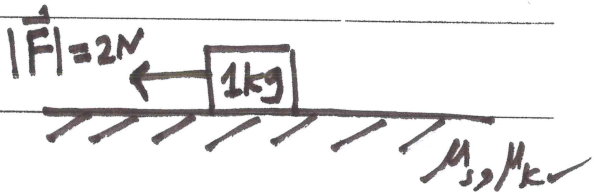
(vi). Equations (1) and (2) are not vector equations! Because f and n are perpendicular to each other, the vectors can NOT be related by a multiplicative constant. They are relationships between the magnitudes of the vectors representing the friction and normal forces.

[II] Applications of the friction force.

Knowing about static and kinetic friction allows us to approximate real-world situations and come to meaningful conclusions.

[1] A typical exam problem; you better watch out...!

A 1 kg block rests on a rough horizontal surface. Then, a pull force of magnitude 2 N is applied to the block as shown. Knowing that $\mu_s = 0.3$ and $\mu_k = 0.1$, what is the friction force, $\vec{f}$, acting on the block? Take $g = 10\,m/s^2$?

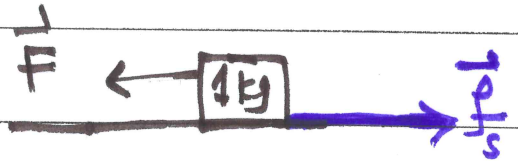


Eq. (1) reads: $f_{s\,max} = \mu_s n$

$f_{s\,max} = \mu_s mg = 3N$.

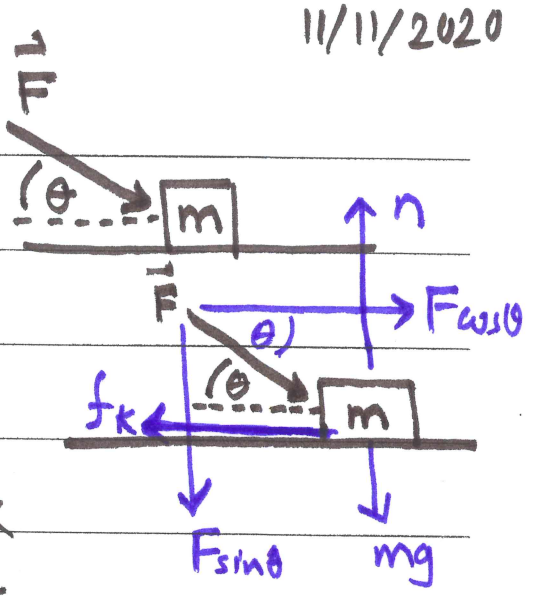
Thus $|F|$ is less than the threshold, $f_{s\,max}$, needed to set the block into motion. In that case, according to Newton's 2nd law, $\vec{f}$ is $\vec{f}_s$ not $\vec{f}_{s\,max}$ neither $\vec{f}_k$

$\therefore \vec{f} = 2N$ to the right



* Rationally, argue why the answers $|\vec{f}| = 3N$ and $|\vec{f}| = 1N$ are incorrect!

2 Past exam:



As shown, the block m is pushed across a horizontal surface by the force $\vec{F}$. Knowing that the coefficient of kinetic friction between the block and the rough surface is 0.30, what is the magnitude of the acceleration of the block?

Given: $m = 3\,kg$, $|\vec{F}| = 20\,N$, $\theta = 30^\circ$.

$\Rightarrow$ Newton's 2nd law: $F\cos\theta - f_k = ma$, (recall item (v) page 3)

$\Rightarrow$ Eq (2): $f_k = \mu_k n \Rightarrow n \neq mg$!!, $n = mg + F\sin\theta$

$\therefore a = \frac{1}{m}\left[F\cos\theta - \mu_k(mg + F\sin\theta)\right]$ ——— (*)

$\Rightarrow$ Inserting the numbers given in the problem statement, we obtain $a = 1.79\,m/s^2$.

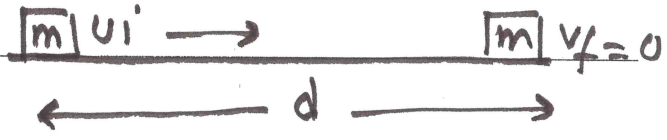
- Validation: What does eq (*) reduce to if the angle θ goes to zero? Does this algebraic expression match your intuition?

- Compare the results of ex 4.18 and ex 4.11. 5/16
- Do ex 4.17.

3 Past exam: A mass $m = 0.5\,kg$ on a frozen pond is given an initial speed of $20\,m\,s^{-1}$. If the mass always remains on the ice and slides 115 m before coming to rest, determine the coefficient of kinetic friction between the mass and ice.

Recall item (v) page 3: f_k slows the mass, which eventually comes to rest due to that force.

Imagine the mass slides to the right, then f_k acts to the left



$\therefore \quad -f_k = ma \;,\; n = mg \Rightarrow -\mu_k mg = ma$

$\therefore \quad a = -\mu_k g \quad — (1)$

Because the velocity of the mass is to the right, the mass is slowing down: the -ve sign means that a is to the left. $\Rightarrow$ a is constant, $\Rightarrow$

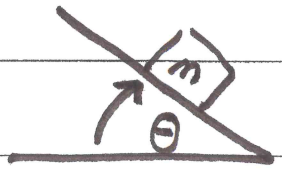
$$v_f^2 = v_i^2 + 2\vec{a}\Delta\vec{x} \Rightarrow 0 = v_i^2 + 2(-\mu_k g)(+d)$$

$$\therefore \quad \mu_k = \frac{v_i^2}{2gd} \quad — (2) \quad = 0.177$$

6/16

- Notice that μ_k (eq 2) is dimensionless, as it should be, and it has a low value, consistent with an object sliding on ice.

[4] Experimental determination of μ_s.

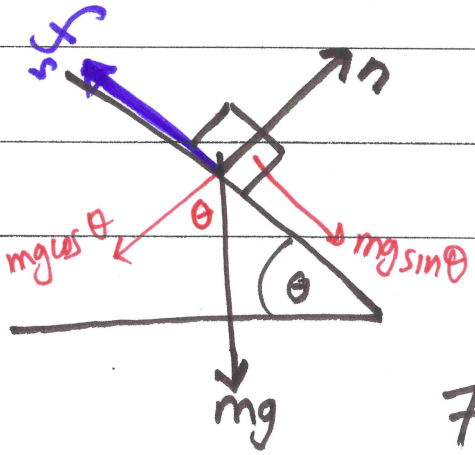


Suppose a block m is placed on a rough surface inclined relative to the horizontal as shown. The incline angle θ is increased until the block starts to move. Determine μ_s by measuring the critical angle θ_c at which the block is just <u>on the verge of slipping</u>.

⇒ Notice how this problem differs from problem [8], lecture 5. When there is no friction on an incline, <u>any</u> angle of the incline will cause a stationary block to begin moving. When there is friction, however, there is <u>no</u> motion for angles less than the critical angle θ_c. θ_c is the angle at which the block is just ready to begin to move <u>but is not moving</u>.

$$mg\sin\theta = f_s \quad \text{---} \quad (1)$$

$$n = mg\cos\theta \quad \text{---} \quad (2)$$



When the incline angle is increased until it equals θ_c, the force f_s has reached its maximum value $f_{s,max}$, i.e., as $\theta = \theta_c \Rightarrow f_s = f_{s,max}$ — (3).

$\Rightarrow$ Eq (1) reads: $mg\sin\theta_c = f_{s,max} = \mu_s n$ — (*)

$\Rightarrow$ Substitute $n = mg\cos\theta$ from eq (2) into (*):

$$mg\sin\theta_c = \mu_s mg\cos\theta_c \Rightarrow \boxed{\mu_s = \tan\theta_c} \quad \text{— (4)}$$

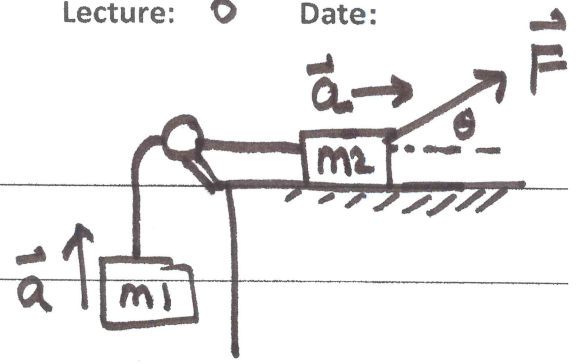
Thus, we have shown that μ_s is related only to θ_c.

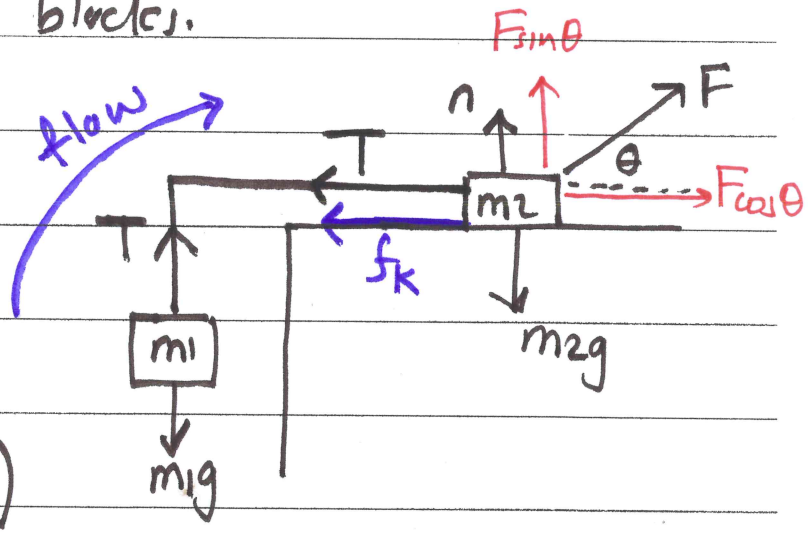
- For example, if the block just slips at $\theta_c = 20^\circ$, then $\mu_s = 0.364$.
- Once the block starts to slip at $\theta \geq \theta_c$, it accelerates down the incline and the force of friction is $f_k = \mu_k n$.
- Do problem 4.38 and ex 4.21.
- Past exam: For the same incline, take $m = 13\,kg$, θ of the incline $= 14^\circ$, the block moves down the incline with initial speed of $1.4\,m\,s^{-1}$, $\mu_k = 0.38$. What is the acceleration of the block?

$$a = g[\sin\theta - \mu_k\cos\theta] = -1.24\,m\,s^{-2}.$$

- Quiz: If the block moves down the incline with constant speed, then $a = g[\sin\theta - \mu_k\cos\theta] = 0 \Rightarrow \boxed{\mu_k = \tan\theta}$.

5 A block of mass m_2 on a rough, horizontal surface is connected to a block of mass m_1 by a cord over a pulley as shown. A force of magnitude F at an angle θ with the horizontal is applied to m_2, and the block slides to the right. The coefficient of kinetic friction between m_2 and surface is μ_k. Determine the magnitude of the acceleration of the two blocks.





$F\cos\theta - f_k - T = m_2 a$ — ①

$T - m_1 g = m_1 a$ — ②

$n = m_2 g - F\sin\theta$ — ③

$\Rightarrow f_k = \mu_k n = \mu_k (m_2 g - F\sin\theta)$

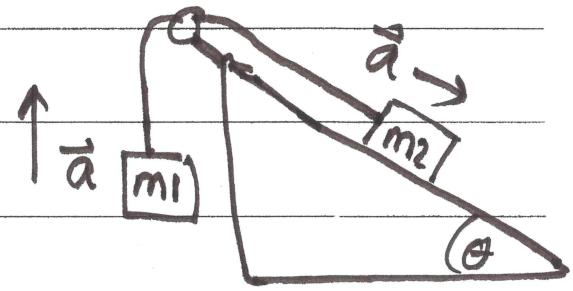
Substitute f_k and the value of T from eq ② into eq ① and solve for a:

$$a = \frac{F(\cos\theta + \mu_k \sin\theta) - (m_1 + \mu_k m_2) g}{(m_1 + m_2)} \quad — ④$$

※ Let's elaborate on problem [5]:

(1) What does eq (4) reduce to if the force $\vec{F}$ is removed and the surface becomes frictionless? Call this expression eq (5). Does this algebraic expression match your intuition about the physical situation in this case?

(2) Now solve for the acceleration of the two blocks for the following problem:



The two blocks m_1, m_2 are attached by a cord that passes over a pulley. Block m_2 lies on a frictionless incline of angle θ. Find the magnitude of the acceleration of the two blocks. Call this expression eq (6).

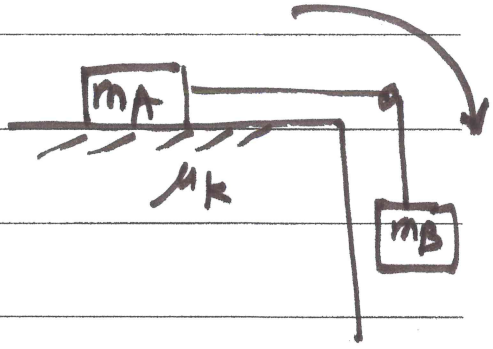
(3) Now let angle θ go to zero in eq (6). How does the resulting equation compare with your equation (5) here in problem [5]? 10/16

(4) Problem (2) above becomes the Atwood machine of problem [7] lecture 5, by letting $\theta \to 90^\circ$ in eq (6) above. Verify that eq (6) above reduces to the result of page 7 lecture 5.

6 Problem 4.44, Problem 4.45 and Example 4.20:

(i) ▪ Read example 4.20:

Solve for the acceleration a:



$$m_B g - T = m_B a$$

$$T - \mu_k m_A g = m_A a$$

$$\Rightarrow \quad a = \left(\frac{m_B - \mu_k m_A}{m_A + m_B}\right) g \quad \text{—— (1)}$$

▪ Compare your result with that of problem (7) lecture 5.

▪ If $m_B \gg m_A \Rightarrow a = g \Rightarrow m_B$ is in free fall ✓.

(ii) ▪ Problem 4.44: For the same system: "what is m_A needed to prevent any motion from occurring?"

$\Rightarrow$ We need μ_s not μ_k and set a of (1) $= 0$

$\Rightarrow$ then $m_B = \mu_s m_A \Rightarrow m_A = \frac{m_B}{\mu_s} = \frac{2}{0.3}$ kg.

(iii) ▪ Problem 4.45: For the same system:

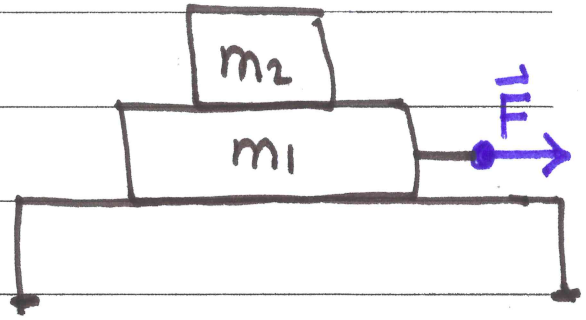
(a) What is m_A needed to keep the system from starting to move?" $\Rightarrow$ set a of (1) $= 0 \rightarrow m_A = \frac{m_B}{\mu_s} = \frac{2}{0.4}$ kg.

(b) "What is m_A needed to keep the system moving at constant speed?" $\Rightarrow$ set a of (1) $= 0$

$\rightarrow m_A = \frac{m_B}{\mu_k} = \frac{2}{0.2}$ kg

7 A "challenge" problem: Two stacked blocks:

Consider two stacked blocks on a table as shown. The upper block has a mass m_2, and the lower block has a mass m_1.



The coefficient of kinetic friction between m_1 and the table is μ_k. The coefficient of static friction between the blocks is μ_s. A string is attached to m_1, and an external force $\vec{F}$ is applied horizontally, pulling on the string as shown.

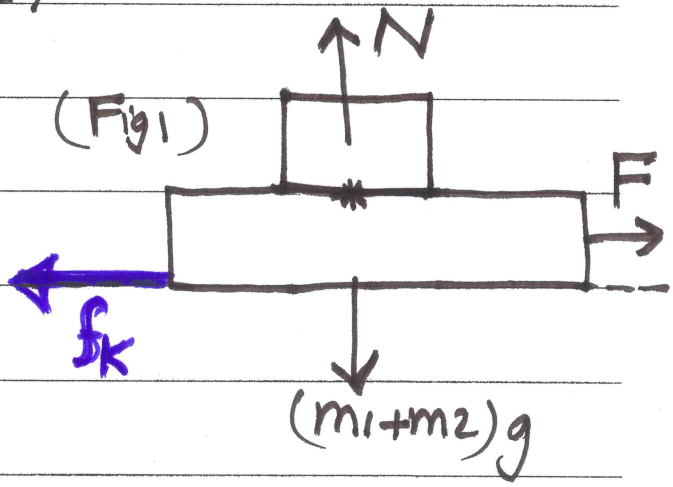
■ What is the maximum force that can be applied to the string without having the upper block slide off?

(1) Notice that as long as the force of static friction between the two blocks is not overcome, the two blocks will travel together. Thus, if we pull gently on m_1, m_2 will stay in place on top of it, and

the two blocks will slide as one. However, if we pull hard on m_1, the force of static friction between the blocks will not be sufficient to keep m_2 in place and it will begin to slide off m_1.

(2) Let us start by treating the two blocks as one system moving together. The forces acting in this case are the external force F pulling on the string, the force of kinetic friction f_k between m_1 and the surface on which the blocks are sliding, the weight $m_1 g$ of the lower block, the weight $m_2 g$ of the upper block and the normal force exerted by the table (as shown in Figure 1):

(Fig 1)



■ Please read page 2 of this lecture:
f_k is always opposite to the direction of motion of the object relative to the surface it moves on (table).
∴ f_k is indeed to the left.

$$\Rightarrow F - f_k = (m_1+m_2)\,a \quad \text{——— (1)}$$

$$N = (m_1+m_2)\,g \quad \text{——— (2)}$$

$$f_k = \mu_k N = \mu_k (m_1+m_2)\,g \quad \text{——— (3)}$$

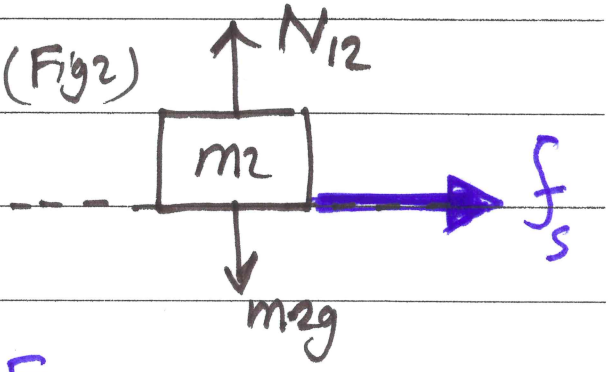
$$\therefore F = (m_1+m_2)(a+\mu_k g) \quad \text{——— (4)}$$

Eq (4) shows that the maximum force is achieved by determining the maximum acceleration a.

(3) Thus, we need to analyze the forces acting on the upper block m_2. The forces for m_2 are the normal force exerted by the lower block N_{12}, the weight $m_2 g$, and the force of static friction f_s (as shown in Figure 2):

- Please "ponder" the direction of f_s: In page 1 of this lecture we said that the direction of $\vec{f}_s$ is always opposite to that of F relative to the surface of contact between m_2 and its supporting surface. This relative surface is **NOT** the surface of the table, but rather the surface of m_1.

(Fig 2)



$\Rightarrow f_s = m_2 a$ —— (5)

$N_{12} = m_2 g$ —— (6)

The maximum value of f_s between the upper and lower blocks is given by: $f_{s\,max} = \mu_s N_{12} = \mu_s m_2 g$. Thus, the maximum acceleration that the upper block can have without sliding is given by eq (5)

$$a_{max} = \frac{f_{s\,max}}{m_2} = \frac{\mu_s m_2 g}{m_2} \Rightarrow a_{max} = \mu_s g \quad \text{—— (7)}$$

(4) This maximum acceleration for m_2 is also the maximum acceleration for both blocks together. We can now relate the maximum acceleration to the maximum force, F_{max}, that can be exerted without the upper block sliding off using eq (7) and eq (4):

$$\therefore F_{max} = (m_1 + m_2)(\mu_s g + \mu_k g)$$

$$\therefore \boxed{F_{max} = (\mu_k + \mu_s)(m_1 + m_2) g} \quad \text{—— (8)}$$

(5) Validation: If there were no friction between the lower block m_1 and the surface of the table, the force required to accelerate both blocks, without the upper one sliding off would be:

recall eq (1) $\Rightarrow$ $F = (m_1 + m_2)a$

recall eq. (7) $a_{max} = \mu_s g$

$\therefore$ $F_{max} = (m_1 + m_2)\mu_s g$ _____ (9)

However, this result could be obtained directly by letting $\mu_k = 0$ in eq (8) ✓.

Thus, our answer (8) for the maximum force seems reasonable because it is higher than the force (9) calculated when there is no friction!